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Robust Missile Feedback Control Strategies

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The nature of the missile interception problem with two noncooperative players leads to the field of differential games. Introducing nonlinear uncertain models can potentially lead to nonexistence of game theoretic equilibrium strategies, computationally intractable problems, and/or highly reduced performance of the strategies. The objective is therefore to obtain robust strategies that provide an upper bound on the performance of the nonlinear system against all allowable disturbances and model uncertainties. Robust programming and more specific robust linear matrix inequality techniques have proved to be an extremely effective tool to design such robust control strategies. This paper extends the existing robust disturbance-feedback framework such that it can effectively deal with a large class of model mismatches and uncertainties such as target maneuvers and nonlinearities. The implementation of these strategies in a receding-horizon fashion on missile intercept problems leads to very satisfactory results when compared to traditional guidance laws.

I. Introduction

THE synthesis of most missile guidance laws is based on nominal models. Generally, uncertainties other than target maneuvers are not taken into account and robustness is verified a posteriori by Monte Carlo simulations [1,2]. Only few examples exist in the missile literature where uncertainties are explicitly considered in the controller synthesis. For example, Yaesh and Ben-Asher [3] have introduced a norm-bounded (scalar) time-variant parametric uncertainty in order to model the variation in the time constant of the transfer function from commanded to actual missile acceleration. Shinar and Shima [4] have designed a robust controller for when noise-corrupted, hence uncertain, measurements of the target's acceleration are available.

The developments in robust model predictive control have renewed the interest in robust control strategies for uncertain discrete-time systems as described in the survey paper by Mayne et al. [5] and references therein. Robust programming can effectively deal with a whole range of different types of uncertainties like real parametric and dynamic uncertainties. Most robust receding-horizon control (RHC) applications implement open-loop min-max strategies to ensure system performance and/or stability. Robust strategies, in pure open-loop form, turn out to be rather conservative and might result in poor guaranteed performance and passive control inputs. This is not surprising, since the open-loop control must satisfy the state and control constraints against all allowable uncertainties and disturbances [6]. On the other hand, the benefits of feedback control are well established. However, incorporating state feedback into robust RHC problems is rather complicated due to nonconvexity issues. In the literature another interesting approach can be found: namely, the application of robust disturbance-feedback strategies [6–8]. These strategies do lead to convex problems and are, equivalent to robust state-feedback strategies in some cases [9]. Robust receding-horizon strategies, based on a robust control design approach developed by Lofberg [6], can be computed by solving linear matrix inequalities (LMIs). In addition to the fact that there exist effective algorithms to solve LMIs, this approach has the additional advantage that a large range of uncertainties can be tackled.

The drawback of the robust disturbance-feedback paradigm developed in previous work is its inability to deal explicitly with system models that have a rational dependency on parametric uncertainties and nonlinearities. Hence, the robust disturbance-feedback framework has, so far, been restricted to disturbances that affect the system additively. The major contribution of this paper is to extend the framework to explicitly include models with rational uncertainty dependency and sector-bounded nonlinearities. The problem to obtain the robust disturbance-feedback strategies is often tackled by using Lagrange relaxation methods [6]. The relaxation of the reformulated problem can potentially lead to conservatism. A remedy against this conservatism is the introduction of a relaxation techniques based on the S-procedure [10]. This relaxation technique is called the full-block S-procedure [11,12] and increases the design flexibility by allowing for a tradeoff between performance and computational load.

The paper starts with a description of a missile intercept to motivate the robust approach described in the sequel of the paper. In Sec. III the problem definition including the objective function, the lifted system and the control strategy, as well as the state, the control and the uncertainty sets are given. The subsequent section briefly describes the full-block S-procedure and the dualization of the problem. In Sec. V the LMIs required to solve the disturbance-feedback problem and the receding-horizon implementation are briefly discussed. In Sec. VI the proposed approach is demonstrated on a simple intercept problem by simulating different intercept scenarios and comparing the performance of the robust disturbance-feedback strategies with different conventional guidance laws. Finally, concluding remarks are presented.

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II. Motivation for Missile Intercept

The role of the missile intercept example discussed in this section is twofold. First, it illustrates and motivates the relevance of a robust design approach for missile intercept guidance. Second, it acts as a test case in Sec. VI to evaluate the performance of the robust approach compared with more conventional guidance laws.

A common approach applied when designing controllers is to neglect dynamics and uncertainties for the sake of simplicity. In this example different uncertainties are added, or more correctly not neglected, to a fourth-order model often used in the literature [3,4,13]. The main objective for the missile is to minimize the final miss distance and restrict the total control effort and/or deviations from its reference trajectory: in this case, the collision course. In Fig. 1 a schematic view of the intercept geometry is shown. The system can be linearized around the initial line of sight λ_0 in the case of near-head-on or tail-chase intercepts. The variables indicated in Fig. 1 are the actual lateral acceleration of the interceptor a_m and the target a_t, the relative lateral distance r and velocity v, all of which are related by the following differential equations:

$$\frac{\mathrm{d}r}{\mathrm{d}t}(t) = v(t) \tag{1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = a_m(t) + a_t(t) \tag{2}$$

The target lateral acceleration a_t is modeled as the output of a first-order system with the commanded acceleration $a_{t,c}$ as its input. Furthermore, it is assumed that the intercept duration is fixed and the uncertain target time constant is captured by a nominal constant τ_e and a polytopic additive uncertainty $\delta_e(\cdot)$, resulting in the following evader dynamics:

$$\frac{\mathrm{d}a_t}{\mathrm{d}t}(t) = \left(\frac{1}{\tau_e} + \delta_e(t)\right)(a_{t,c}(t) - a_t(t)) \tag{3}$$

The interceptor's lateral acceleration a_m is generated by a fin deflection ϕ. It is assumed that increments in fin deflections are less effective when the deflections are larger. This model nonlinearity is expressed as

$$a_m(t) = c(\phi(t)) \quad \text{with} \quad c(\phi) = c_0\phi + c_1\phi^3 \tag{4}$$

Previous examples have considered the actuator dynamics from commanded fin deflection ϕ_c to ϕ as a first-order system. Modeling the actuator dynamics as a second-order system, with damping ratio ζ and natural frequency ω, results in a more accurate description of the dynamics. The mismatch between the actual actuator dynamics and its model can be covered by an additional multiplicative uncertainty δ_ω of ω. The uncertainty δ_ω is assumed to be time-variant such that

$$\begin{aligned}\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}(t) &= -2\zeta(1+\delta_\omega(t))\omega\frac{\mathrm{d}\phi}{\mathrm{d}t}(t) \\ &\quad + (1+\delta_\omega(t))^2\omega^2(-\phi(t)+\phi_c(t))\end{aligned} \tag{5}$$

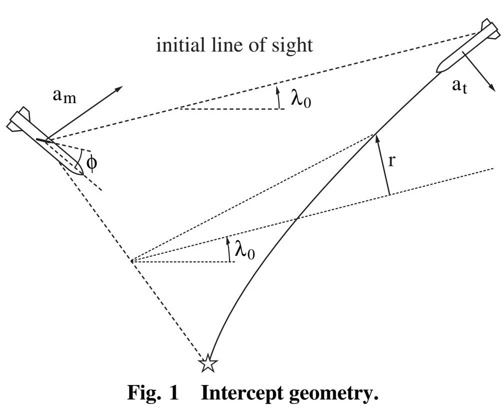


Fig. 1 Intercept geometry.

This differential equation has a polynomial, and therefore rational, dependency on δ_ω. Hence, the intercept systems contain disturbances $a_{t,c}$ and polytopic system uncertainty δ_e that enter the system additively in Eq. (3), rational system uncertainty δ_ω in Eq. (5), and static nonlinearities in Eq. (4).

III. Problem Definition

Consider a robust control problem in which the main goal is to design a controller that minimizes an objective function against worst-case disturbances and uncertainties. Assume that the underlying system has the following, possibly nonlinear, discrete state-space representation:

$$x(k+1) = f(x(k), u(k), d(k), \delta_s(k)), \qquad x(0) = x_0 \tag{6}$$

with $x(k) \in \mathcal{X}(k) \subset \mathbb{R}^n$, $u(k) \in \mathcal{U}(k) \subset \mathbb{R}^{n_u}$, $d(k) \in \mathcal{D}_d(k) \subset \mathbb{R}^{n_d}$, and $\delta_s(k) \in \mathcal{D}_s(k) \subset \mathbb{R}^{n_\delta}$, respectively, denoting the state, the control input, the exogenous additive disturbance and the system uncertainty. The problem has a finite time horizon n_c, where $\mathcal{K} = \{0, \ldots, n_c\}$ is the ordered set of integers indicating the problem's time interval. The uncertainty sets $\mathcal{D}_d(k)$, $\mathcal{D}_s(k)$ and the control constraint set $\mathcal{U}(k)$ are ellipsoids and assumed to be compact for all $k \in \mathcal{K}/\{n_c\}$. The state set $\mathcal{X}(k)$ is ellipsoidal as well. The vector-valued mapping f in Eq. (6) defining the system dynamics is assumed to have the form

$$f(x, u, d, \delta_s) = A(\delta_s)x + B(\delta_s)u + G_d(\delta_s)d + g(x, u) \tag{7}$$

The nonlinear mapping $g\colon \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^n$ has the property $g(0,0) = 0$ and $A(\delta)$, $B(\delta)$ are rational matrix-valued functions of δ without a pole at zero. The model defined by Eq. (7) covers a wide variety of systems encountered in real life and, more specifically, in most missile intercept problems found in the literature [1,2,4,13]. Note that for notational reasons, it is assumed that the system is time-invariant. However, the presented work in this paper can easily be extended to time-varying systems.

A. Objective Function

The control objective is to minimize a function that penalizes control inputs and state deviations from a reference zero-trajectory, while the disturbances and uncertainties can be considered as (worst-case) maximizers. The controller performance, with initial state $x(0) = x_0$, is quantified by the real-valued objective functional J_n as

$$\begin{aligned}J_n(x_0, u(\cdot), d(\cdot), \delta_s(\cdot)) = \sum_{k=0}^{n_c-1} x(k+1)^T Q(k+1) x(k+1) \\ + u(k)^T R(k) u(k)\end{aligned}$$

with the weight matrices satisfying $Q(k+1) \in \mathbb{S}_+^n$ and $R(k) \in \mathbb{S}_{++}^{n_u}$ for all $k \in \mathcal{K}/\{n_c\}$.

In classical min-max games, conditions for the existence of so-called zero-sum saddle-point strategies [14] are derived that guarantee a certain payoff to both players. Hence, one wants to find strategies such that

$$\begin{aligned}&\max_{d(\cdot)\in\mathcal{D}_d, \delta_s(\cdot)\in\mathcal{D}_s} \min_{u(\cdot)\in\mathcal{U}} J_n(x_0, u(\cdot), d(\cdot), \delta_s(\cdot)) \\ &= \min_{u(\cdot)\in\mathcal{U}} \max_{d(\cdot)\in\mathcal{D}_d, \delta_s(\cdot)\in\mathcal{D}_s} J_n(x_0, u(\cdot), d(\cdot), \delta_s(\cdot))\end{aligned}$$

Note that the time argument of the time-varying sets will be omitted when indicating the Cartesian product of the sets, such as in $\mathcal{D}_d = \mathcal{D}_d(0) \times \cdots \times \mathcal{D}_d(n_c - 1)$. Furthermore, the disturbances and uncertainties are not penalized in the objective function, as in most game theoretic literature [14], since both are already contained in compact sets. The saddle-point strategies, denoted with superscript $^\circ$, are obtained under the nonrealistic worst-case assumption that all uncertainties are cooperating and playing as maximizers. These strategies are characterized by the following property:

$$J_n(x_0, u^\circ(\cdot), d(\cdot), \delta_s(\cdot)) \le J_n(x_0, u^\circ(\cdot), d^\circ(\cdot), \delta_s{}^\circ(\cdot)) \le J_n(x_0, u(\cdot), d^\circ(\cdot), \delta_s{}^\circ(\cdot)) \tag{8}$$

Saddle-point strategies do not always exist and are, in many cases, hard to derive. The real interest seen from the controller's perspective is the achieved guaranteed cost expressed in the first inequality in Eq. (8). The value for $J_n(x_0, u^\circ(\cdot), d^\circ(\cdot), \delta_s{}^\circ(\cdot))$ bounds the objective function from above against all allowable uncertainties. If the initial state x_0 and the optimal strategy $u^\circ(\cdot)$ are given, this value can be determined by taking the supremum over all possible uncertainties. However, the supremum is difficult to calculate. For this reason the problem is replaced by a convex problem through a Lagrange relaxation method. The convex problem is defined such that it provides an upper bound $J_u(x_0, u(\cdot))$ for all $u(\cdot) \in \mathcal{U}$ and for all $x_0 \in \mathbb{R}^n$. More precisely, $J_u(x_0, u(\cdot))$ is determined to guarantee that

$$J_n(x_0, u(\cdot), d(\cdot), \delta_s(\cdot)) \le J_u(x_0, u(\cdot)) \quad \forall\ d(\cdot) \in \mathcal{D}_d, \delta_s(\cdot) \in \mathcal{D}_s \tag{9}$$

The problem objective is now to find a strategy $u^*(\cdot) \in \mathcal{U}$ that minimizes $J_u(x_0, u^*(\cdot))$ in order to provide a guaranteed cost against all allowable disturbances and uncertainties. The price paid for replacing the maximization by the convex relaxation problem is a potential relaxation gap, i.e., a possibly conservative upper bound. A description of the developed semidefinite programming (SDP) techniques to obtain the optimal robust strategy and upper bound value is given in Sec. IV.

B. Linear System Representation

In general, the complexity of the controller synthesis is highly reduced when the system is described by linear state equations. In real life it is very rare that a linear model exactly matches the system of interest. A common approach to remain in a linear framework is to approximate the system by a linear nominal model while adding some additional uncertainty to account for the model mismatch. The synthesis problem is then rephrased as a robust synthesis problem. The different uncertainties considered in this study are additive exogenous inputs, real parametric and sector-bounded nonlinear uncertainties. These uncertainties cover a large part of the model uncertainties described in the literature.

It is well known that systems with a rational uncertainty dependency admit a linear fractional representation (LFR) [15]. The motivation to represent the system by an LFR is to remove the rational uncertainty dependency and replace it by a linear dependency in some auxiliary vectors. In our case an LFR of system (6) is given as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + G_d w_d(k) + G_s w_s(k) + G_n w_n(k) \\ z_d(k) &= \mathbf{1}, \qquad z_n(k) = C_n x(k) + D_{nu} u(k) \\ z_s(k) &= C_s x(k) + D_{su} u(k) + D_{sd} w_d(k) + D_{ss} w_s(k) + D_{sn} w_n(k) \\ w_d(k) &= \Delta_d(d(k)) z_d(k), \qquad w_n(k) = h(z_n(k)) \\ w_s(k) &= \Delta_s(\delta_s(k)) z_s(k) \end{aligned} \tag{10}$$

The introduced delta blocks $\Delta_d(d) = \mathrm{diag}(d_1, \ldots, d_{n_u})$ and $\Delta_s(\delta_s) = \mathrm{diag}(\delta_{s,1} I, \ldots, \delta_{s,n_\delta} I)$ are linear matrix-valued mappings of, respectively, d and δ_s. The vector-valued mapping h: $\mathbb{R}^{n_j} \to \mathbb{R}^{n_j}$ has the property that $G_n h(C_n x + D_{nu} u) = g(x, u)$. Note that boldface $\mathbf{1}$ denotes the all-ones vector.

C. Uncertainty Sets

The assumption that $\mathcal{D}_d(k)$, $\mathcal{D}_s(k)$, $\mathcal{U}(k)$ and $\mathcal{X}(k)$ are compact ellipsoids is not really restrictive, since it allows for a large variety of sets such as polytopes, slabs, and cylinders [16]. Furthermore, stochastic uncertainty can, in a rather straightforward way, be modeled as a deterministic uncertainty contained in an ellipsoid. The most important justification, however, is the easy manipulations of ellipsoids and their simple parametric representation.

Suppose that the nonlinear mapping $g(x, u)$ only depends on n_l components of the vector x, which are stacked in a vector $x_l \in \mathbb{R}^{n_l}$. This vector x_l is an element of $\mathcal{X}_l(k) \triangleq \{x_l \in \mathbb{R}^{n_l} | x \in \mathcal{X}(k)\}$ for all $k \in \mathcal{K}$. The sets $\mathcal{X}_l(k)$ are assumed ellipsoidal and compact. One of the implications of this assumption is that

$$\mathcal{Z}_n(k) \triangleq \{C_n x + D_{nu} u | x_l \in \mathcal{X}_l(k), u \in \mathcal{U}(k)\}$$

is convex and compact, since $\mathcal{Z}_n(k)$ is a linear image of the compact set $\mathcal{X}_l(k) \times \mathcal{U}(k)$.

Recall that in our case $g(x, u) = G_n w_n$, $w_n = h(z_n)$ and $z_n = C_n x + D_{nu} u$. For our purpose, these static nonlinearities can be considered as arbitrary fast-varying uncertainties that satisfy box constraints (see Fig. 2) and enter the system additively. The major drawback of this approach is the negligence of the structure in the uncertainty set, which leads to conservatism. A potential remedy against the conservatism is to work with sector constraints. In this study only vector-valued mappings h are considered that can be decomposed into n_j real-valued mappings h_j: $\mathbb{R} \to \mathbb{R}$ as $h(z_n) = \mathrm{col}(h_1(z_{n,1}), \ldots, h_{n_j}(z_{n,n_j}))$. The pairs $(w_{n,j}, z_{n,j})$ with $w_{n,j} = h_j(z_{n,j})$ are said to satisfy a sector constraint for all $j = 1, \ldots, n_j$ if they are contained in a conic sector (see Fig. 2), which can also be expressed as

$$w_{n,j} = h_j(z_{n,j}) \in \{\delta_{n,j} z_{n,j} | \delta_{n,j} \in [\beta_j, \sigma_j]\}$$

Diagonal augmentation of all the variables $\delta_{n,j}$ as $\Delta_n(\delta_n) = \mathrm{diag}(\delta_{n,1}, \ldots, \delta_{n,n_j})$ results in the existence of $\delta_n \in \{\delta_n \in \mathbb{R}^{n_j} | \delta_{n,j} \in [\beta_j, \sigma_j], j = 1, \ldots, n_j\}$ such that $h(z_n) = \Delta_n(\delta_n) z_n$. This approach of covering the nonlinearities is taken from Boyd et al. [17].

One can now impose constraints on the delta blocks such that, for all $k \in \mathcal{K}/\{n_c\}$, the condition $d \in \mathcal{D}_d(k)$ implies $\Delta_d(d) \in \mathcal{S}_d(k)$ and $\delta_s \in \mathcal{D}_s(k)$ implies $\Delta_s(\delta_s) \in \mathcal{S}_s(k)$; furthermore,

$$h_j(z_{n,j}) \in \{\delta_{n,j} z_{n,j} | \delta_{n,j} \in [\beta_j, \sigma_j]\} \quad \forall\ z_{n,j} \in \mathcal{Z}_{n,j}(k), \quad j = 1, \ldots, n_j$$

implies that $\Delta_n(\delta_n) \in \mathcal{S}_n(k)$.

D. Lifted System

The LFR of the dynamical system (6) with finite horizon n_c can be replaced by a lifted system. The vector $\mathbf{w}_d$ is defined as $\mathbf{w}_d = \mathrm{col}(w_d(0), \ldots, w_d(n_c - 1))$ and $\mathbf{w}_s$, $\mathbf{w}_n$ are obtained similarly. The operation $\mathrm{col}(x_1, \ldots, x_n)$ stacks the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in one vector. The lifted states $\mathbf{x} = \mathrm{col}(x(0), \ldots, x(n_c))$ are obtained by a linear vector-valued mapping of the initial state x_0, the control input $\mathbf{u} = \mathrm{col}(u(0), \ldots, u(n_c - 1))$ and the vector $\mathbf{w} = \mathrm{col}(\mathbf{w}_d, \mathbf{w}_s, \mathbf{w}_n)$ defined as

$$\mathbf{x} = f_x(x_0, \mathbf{u}, \mathbf{w}) = \mathbf{A}_0 x_0 + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \tag{11}$$

with $\mathbf{G} = [\,\mathbf{G}_d \quad \mathbf{G}_s \quad \mathbf{G}_n\,]$ and

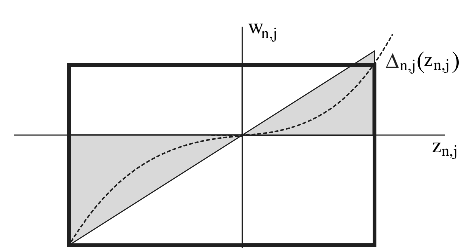


Fig. 2 Box vs sector constraints.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ I \otimes A & 0 \end{bmatrix}, \qquad \mathbf{B} = (I - \mathbf{A})^{-1} \begin{bmatrix} 0 \\ I \otimes B \end{bmatrix}$$

$$\mathbf{A}_0 = \begin{bmatrix} I \\ A \\ \vdots \\ A^{n_c - 1} \end{bmatrix}, \qquad \mathbf{G}_i = (I - \mathbf{A})^{-1} \begin{bmatrix} 0 \\ I \otimes G_i \end{bmatrix}, \qquad i = d, s, n$$

Note that boldface symbols are used to refer to the lifted vectors and stacked matrices, and $\otimes$ denotes the Kronecker product. The objective function, which is quadratic in the state and control, in lifted form is

$$J(x_0, \mathbf{u}, \mathbf{w}) = \mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} \tag{12}$$

where $\mathbf{Q} = \mathrm{diag}(Q(1), \ldots, Q(n_c))$ and $\mathbf{R} = \mathrm{diag}(R(0), \ldots, R(n_c - 1))$. The objective function can be expressed as $J(x_0, \mathbf{u}, \mathbf{w}) = \mathbf{z}_p^T \mathbf{z}_p$ if the controlled output vector $\mathbf{z}_p$ is given as

$$\mathbf{z}_p = \begin{bmatrix} \mathbf{Q}^{1/2} \mathbf{x} \\ \mathbf{R}^{1/2} \mathbf{u} \end{bmatrix} \tag{13}$$

The following lifted auxiliary vectors

$$\begin{gathered} \mathbf{w}_d = \mathbf{\Delta}_{ld}(\mathbf{d}) \mathbf{z}_d, \qquad \mathbf{w}_n = \mathbf{\Delta}_n(\boldsymbol{\delta}_n) \mathbf{z}_n, \qquad \mathbf{w}_s = \mathbf{\Delta}_s(\boldsymbol{\delta}_s) \mathbf{z}_s \\ \mathbf{z}_d = \mathbf{1}, \qquad \mathbf{z}_n = \mathbf{C}_n \mathbf{A}_0 x_0 + (\mathbf{C}_n \mathbf{B} + \mathbf{D}_{nu}) \mathbf{u} + \mathbf{C}_n \mathbf{G} \mathbf{w} \\ \mathbf{z}_s = \mathbf{C}_s \mathbf{A}_0 x_0 + (\mathbf{C}_s \mathbf{B} + \mathbf{D}_{su}) \mathbf{u} + (\mathbf{C}_s \mathbf{G} + \mathbf{D}_{sw}) \mathbf{w} \end{gathered} \tag{14}$$

together with Eq. (13) define the LFR of the lifted system. The following abbreviations are used

$$\begin{gathered} \mathbf{C}_i = \begin{bmatrix} I \otimes C_i & 0 \end{bmatrix}, \qquad i = n, s \qquad \mathbf{D}_j = I \otimes D_j \\ j = nu, su, sd, ss, sn \qquad \mathbf{D}_{sw} = [\mathbf{D}_{sd} \quad \mathbf{D}_{ss} \quad \mathbf{D}_{sn}] \end{gathered}$$

Hence, the lifted system is represented by a LFR that maps the initial state x_0, the control input $\mathbf{u}$, the disturbances $\mathbf{d}$ and the uncertainties $\boldsymbol{\delta}_s$ and $\boldsymbol{\delta}_n$ directly to the controlled output vector $\mathbf{z}_p$.

The delta blocks are lifted by diagonal augmentation as $\mathbf{\Delta}_{ld}(\mathbf{d}) = \mathrm{diag}(\Delta_d(d(0)), \ldots, \Delta_d(d(n_c - 1)))$, with a similar procedure applied to $\mathbf{\Delta}_s(\boldsymbol{\delta}_s)$ and $\mathbf{\Delta}_n(\boldsymbol{\delta}_n)$. The operation $\mathrm{diag}(\cdot)$ maps the elements of a vector into a diagonal matrix. The constraints on the delta blocks in lifted form, with $\boldsymbol{\delta} = \mathrm{col}(\mathbf{d}, \boldsymbol{\delta}_s, \boldsymbol{\delta}_n)$ and $\mathbf{\Delta}(\boldsymbol{\delta}) = \mathrm{diag}(\mathbf{\Delta}_{ld}(\mathbf{d}), \mathbf{\Delta}_s(\boldsymbol{\delta}_s), \mathbf{\Delta}_n(\boldsymbol{\delta}_n))$, are expressed as $\mathbf{\Delta} \in \mathcal{S}$. The control and state constraints can be similarly stacked. These constraints are enforced by inequality constraints on, respectively, affine matrix-valued mappings E and F such that

$$E(\mathbf{u}) \preceq 0 \Leftrightarrow \mathbf{u} \in \mathcal{U}, \qquad F(\mathbf{x}) \preceq 0 \Leftrightarrow \mathbf{x} \in \mathcal{X} \tag{15}$$

The notation $A \succ B$ ($A \succeq B$) means that A–B is positive definite (semidefinite) with respect to the semidefinite cone denoted as $\mathbb{S}^m_{++}$ ($\mathbb{S}^m_+$). The controlled output vector $\mathbf{z}_p$ is obtained by using $\mathbf{x} = f_x(x_0, \mathbf{u}, \mathbf{w})$, which, if combined with Eqs. (13) and (14) and $\mathbf{z} = \mathrm{col}(\mathbf{z}_d, \mathbf{z}_s, \mathbf{z}_n)$, finally results in

$$\begin{bmatrix} \mathbf{z}_p \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} A_l(x_0, \mathbf{u}) & B_l \\ C_l(x_0, \mathbf{u}) & D_l \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{w} \end{bmatrix}, \qquad \mathbf{w} = \mathbf{\Delta}(\boldsymbol{\delta}) \mathbf{z} \tag{16}$$

E. State and Control Sets

Affine disturbance-feedback strategies differ from the commonly used open-loop strategies, since they consist of two different terms: namely, disturbance-feedback and open-loop components. Affine disturbance-feedback strategies [6–8] are defined by

$$\mathbf{u} = f_u(\mathbf{v}, \mathbf{M}, \mathbf{d}) = \mathbf{v} + \mathbf{M} \mathbf{d} \tag{17}$$

with the lifted open-loop vector $\mathbf{v} \in \mathbb{R}^{n_c \cdot n_u}$ and the lifted lower block triangular disturbance-feedback gain matrix $\mathbf{M} \in \mathbb{R}^{n_c n_u \times n_c n_d}$ structured as

$$\mathbf{M} = \begin{bmatrix} 0 & & & 0 \\ M_{2,1} & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ M_{n_c,1} & & M_{n_c,n_c-1} & 0 \end{bmatrix}$$

Note that open-loop strategies are included in the set of affine disturbance-feedback strategies if setting $\mathbf{M} = 0$. Furthermore, a disturbance-feedback horizon n_m is introduced such that for $j = 1, \ldots, n_c - 1$ the property $M_{i+j,j} = 0$ holds for all $i > n_m$.

IV. SDP Relaxation Techniques

The robust problem can be reformulated as the minimization of an upper bound γ^2 on the Euclidian norm of the controlled output vector (13) such that the performance upper bound, control, and state constraints are respected for all allowable disturbances and uncertainties. The controlled output vector is obtained by the LFR (16) of the lifted system as

$$\mathbf{z}_p = A_l(x_0, \mathbf{u}) + B_l \mathbf{w}, \qquad \mathbf{z} = C_l(x_0, \mathbf{u}) + D_l \mathbf{w}, \qquad \mathbf{w} = \mathbf{\Delta}(\boldsymbol{\delta}) \mathbf{z}$$

Both mappings [Eqs. (11) and (17)], defining, respectively, the state $\mathbf{x}$ and the control $\mathbf{u}$, are functions of the disturbance and/or uncertainty. Therefore, the constraint on $\mathbf{x}$ and $\mathbf{u}$ [Eq. (15)] needs to be respected for all $\mathbf{\Delta} \in \mathcal{S}$. Hence, the problem can be formulated as

$$\begin{gathered} \min_{\mathbf{u}, \gamma} \quad \gamma \qquad \text{subject to } \mathbf{z}_p^T \mathbf{z}_p < \gamma^2 \quad \forall\ \mathbf{\Delta} \in \mathcal{S} \\ E(\mathbf{u}) \preceq 0 \quad \forall\ \mathbf{\Delta} \in \mathcal{S}, \qquad F(\mathbf{x}) \preceq 0 \quad \forall\ \mathbf{\Delta} \in \mathcal{S} \end{gathered} \tag{18}$$

The direct minimization of the upper bound γ and the robust satisfaction of the constraints is still a difficult problem due to the semi-infinite constraints and, therefore, relaxation techniques are introduced in this section. Interested readers not familiar with relaxation techniques [18], the S-procedure [10], LMI relaxations [19], and, more specifically, the full-block S-procedures [11] are referred to the given references.

A. Full-Block *S*-Procedure

The key technique employed in this paper, to reformulate the computationally intractable problem into a LMI problem, is a Lagrange relaxation technique based on the S-procedure [10]. The relaxed LMI problem can be solved by effective computational algorithms [18], resulting in a robust strategy. The approach is similar to that in the work of Lofberg [6]. However, in the present work, the types of multipliers used are not restricted to scalar-diagonal multiplier, but they are extended to the larger class of full-block multipliers. The following theorem states the full-block S-procedure theorem [11,12]. The theorem holds for Hermitian matrix multipliers; however, in the sequel, the multipliers are restricted to real symmetric matrices in $\mathbb{R}$, denoted by $\mathbb{S}$.

Theorem 1: A performance specification expressed, with Hermitian P, as

$$\begin{bmatrix} I \\ A_l + B_l \mathbf{\Delta}(I - D_l \mathbf{\Delta})^{-1} C_l \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ (P_{12})^T & P_{22} \end{bmatrix} \times \begin{bmatrix} I \\ A_l + B_l \mathbf{\Delta}(I - D_l \mathbf{\Delta})^{-1} C_l \end{bmatrix} \prec 0 \tag{19}$$

holds for all $\mathbf{\Delta} \in \mathcal{S}$ if there exist symmetric multipliers Λ such that

$$\begin{bmatrix} \mathbf{\Delta} \\ I \end{bmatrix}^T \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \mathbf{\Delta} \\ I \end{bmatrix} \succeq 0 \quad \forall \mathbf{\Delta} \in \mathcal{S} \tag{20}$$

and

$$\begin{bmatrix} I & 0 \\ 0 & I \\ A_l & B_l \\ C_l & D_l \end{bmatrix}^T \underbrace{\begin{bmatrix} P_{11} & 0 & P_{12} & 0 \\ 0 & \Lambda_{11} & 0 & \Lambda_{12} \\ (P_{12})^T & 0 & P_{22} & 0 \\ 0 & \Lambda_{12}^T & 0 & \Lambda_{22} \end{bmatrix}} \begin{bmatrix} I & 0 \\ 0 & I \\ A_l & B_l \\ C_l & D_l \end{bmatrix} \prec 0 \tag{21}$$

Furthermore, if the additional constraint $P_{22} \succeq 0$ holds, then Eq. (21) provides a sufficient condition for nonsingularity of $I - D_l\mathbf{\Delta}$ for all $\mathbf{\Delta} \in \mathcal{S}$. If $\mathcal{S}$ is compact, existence of Λ is also necessary.

Proof. See Scherer [11]. □

Hence, the S-procedure transforms the rational semi-infinite uncertainty dependence in the performance specification [Eq. (19)] into a quadratic semi-infinite dependency [Eq. (20)] by introducing the additional multiplier Λ.

B. Full-Block Multipliers

The set of all multipliers Λ that satisfy the quadratic (semi-infinite) inequality (20) can, in many cases, not be expressed tightly by a finite number of LMIs. Therefore, one settles for a potentially more conservative inner approximation of the multiplier set. We introduce a linear matrix-valued mapping S such that the set of all multipliers Λ for which $S(\Lambda)$ is negative-definite is a subset of all multipliers that satisfy Eq. (20). Therefore,

$$S(\Lambda) \prec 0 \Rightarrow \begin{bmatrix} \mathbf{\Delta} \\ I \end{bmatrix}^T \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \mathbf{\Delta} \\ I \end{bmatrix} \succeq 0 \quad \forall \mathbf{\Delta} \in \mathcal{S} \tag{22}$$

The rational behind the introduction of the mapping S is the simplification by the negative-definite condition on $S(\Lambda)$ when compared with the original semi-infinite condition (20).

However, the simplification has the disadvantage that only a subset of all multipliers Λ that satisfy Eq. (20) are considered, which might lead to conservatism. The choice of the mapping S determines the multiplier structure and hence the subset of all multipliers satisfying Eq. (20). If this subset of multipliers becomes larger, the inner approximation of the set of all multipliers is more refined and the conservatism will potentially decrease. The drawback of better inner approximations is the increase in computational complexity.

The different types of multipliers [19] used in this study are block-diagonal and full-block multipliers. Furthermore, it is assumed that $\mathbf{\Delta}(\boldsymbol{\delta})$ consist of real repeated scalar blocks, i.e., $\mathbf{\Delta}(\boldsymbol{\delta}) = \mathrm{diag}(\boldsymbol{\delta}_1 I, \ldots, \boldsymbol{\delta}_{n_q} I)$, and the uncertainty block set $\mathcal{S}$ is defined as $\mathcal{S} = \{\mathbf{\Delta} \in \mathbb{S} \,|\, \|\mathbf{\Delta}\|^2 \leq 1\}$. The block-diagonal multipliers, called DG scalings in μ theory, consist of n_q skew-symmetric subblocks such that $\Lambda_{12} = \mathrm{diag}(\Lambda_{12}(1), \ldots, \Lambda_{12}(n_q))$ and n_q symmetric subblocks such that $\Lambda_{11} = \mathrm{diag}(\Lambda_{11}(1), \ldots, \Lambda_{11}(n_q))$, with $\Lambda_{22} = -\Lambda_{11}$ are defined as

$$S(\Lambda_{\mathrm{dg}}) = \Lambda_{11} \prec 0 \Rightarrow \begin{bmatrix} \mathbf{\Delta} \\ I \end{bmatrix}^T \Lambda_{\mathrm{dg}} \begin{bmatrix} \mathbf{\Delta} \\ I \end{bmatrix} \succeq 0 \quad \forall \mathbf{\Delta} \in \mathcal{S}$$

Substituting these multipliers in condition (20) and exploiting the skew-symmetric multiplier Λ and the real repeated scalar $\mathbf{\Delta}$ structures leads to

$$\mathrm{diag}\left((\boldsymbol{\delta}_1^2 - 1)\Lambda_{11}(1), \ldots, (\boldsymbol{\delta}_{n_q}^2 - 1)\Lambda_{11}(n_q)\right) \succeq 0$$

Now it is trivial to see that for all $\mathbf{\Delta} \in \mathcal{S}$ (hence, $|\boldsymbol{\delta}_j| \leq 1$ for $j = 1, \ldots, n_q$), and with sufficiency condition $\Lambda_{11}(j) \preceq 0$ for $j = 1, \ldots, n_q$, the condition (20) is satisfied and the skew-symmetric multipliers are a subset of all admissible multipliers. In the case of a scalar-diagonal structure of $\mathbf{\Delta}$ the multiplier blocks reduced to a scalar-diagonal matrix Λ_{11} and a zero matrix $\Lambda_{12} = 0$ and exactly matches the multipliers used by Lofberg [6]. The full-block multipliers require that the uncertainties are captured in a convex hull representation defined by n_e extreme points $\mathbf{\Delta}_1, \ldots, \mathbf{\Delta}_{n_e}$; hence, $\mathcal{S} = \mathrm{conv}(\mathbf{\Delta}_1, \ldots, \mathbf{\Delta}_{n_e})$. The full-block multipliers are then chosen as

$$S_0(\Lambda_f) = \Lambda_{11} \prec 0, \qquad S_j(\Lambda_f) = \begin{bmatrix} \mathbf{\Delta}_j \\ I \end{bmatrix}^T \Lambda_f \begin{bmatrix} \mathbf{\Delta}_j \\ I \end{bmatrix} \succeq 0$$

$$j = 1, \ldots, n_e \Rightarrow \begin{bmatrix} \mathbf{\Delta} \\ I \end{bmatrix}^T \Lambda_f \begin{bmatrix} \mathbf{\Delta} \\ I \end{bmatrix} \succeq 0 \quad \forall\, \mathbf{\Delta} \in \mathcal{S}$$

The inequality $\Lambda_{11} \prec 0$ implies that left-hand part of Eq. (20) is concave in $\mathbf{\Delta}$ on $\mathcal{S}$. Hence, if and only if inequality (20) is satisfied at the extreme points of the convex hull $\mathcal{S}$, enforced by the inequalities on the mappings $S_j(\Lambda_f)$ for $j = 1, \ldots, n_q$, it is satisfied for all $\mathbf{\Delta} \in \mathcal{S}$. Note that the full-block multipliers do not require any structure in the $\mathbf{\Delta}$ blocks.

C. Dualization

The full-block S-procedure, Theorem 1, can now be applied to transform problem (18), with its semi-infinite constraints, into a tractable SDP. The performance specification in Eq. (18) can be expressed in the desired S-procedure from Eq. (19) by defining $P_{11} = -\gamma$, $P_{12} = 0$ and $P_{22} = \frac{1}{\gamma} I$. The subblocks $A_l(x_0, \mathbf{u})$ and $C_l(x_0, \mathbf{u})$ of LFR (16) are linear matrix-valued mappings and B_l, D_l are independent of any decision variable in the case of open-loop controls ($\mathbf{M} = 0$). The arguments of $A_l(x_0, \mathbf{u})$ and $C_l(x_0, \mathbf{u})$ will be omitted in the sequel for notational convenience. The SDP conditions resulting from the S-procedure, with a well-defined matrix K in Eq. (21), the property $P_{22} \succeq 0$ and the quadratic inequality (20) on $\mathbf{\Delta}$ can be expressed as

$$\begin{bmatrix} I & 0 \\ 0 & I \\ A_l & B_l \\ C_l & D_l \end{bmatrix}^T K \begin{bmatrix} I & 0 \\ 0 & I \\ A_l & B_l \\ C_l & D_l \end{bmatrix} \prec 0, \quad \begin{bmatrix} 0 & 0 \\ \mathbf{\Delta} & 0 \\ 0 & I \\ I & 0 \end{bmatrix}^T K \begin{bmatrix} 0 & 0 \\ \mathbf{\Delta} & 0 \\ 0 & I \\ I & 0 \end{bmatrix} \succeq 0 \;\; \forall\, \mathbf{\Delta} \in \mathcal{S} \tag{23}$$

It can be observed that this SDP, in the sequel denoted as the primal SDP, is not an LMI, since the SDP is not linear in $A_l(x_0, \mathbf{u})$, $C_l(x_0, \mathbf{u})$, γ, and Λ. The dualization lemma is now introduced as a tool to transform the primal SDP into a more desirable form, which will lead to a LMI problem formulation.

Lemma 2: Let K be a nonsingular matrix in $\mathbb{S}^n$, and let $\mathcal{V}_1$, $\mathcal{V}_2$ be two complementary subspaces whose sum equals $\mathbb{R}^n$. Then

$$x^T K x < 0 \quad \forall\, x \in \mathcal{V}_1\{0\}, \qquad x^T K x \geq 0 \quad \forall\, x \in \mathcal{V}_2$$

is equivalent to

$$x^T K^{-1} x > 0 \quad \forall\, x \in \mathcal{V}_1^{\perp}/\{0\}, \qquad x^T K^{-1} x \leq 0 \quad \forall\, x \in \mathcal{V}_2^{\perp}$$

Proof. See, for example, Scherer [11]. □

The dualization lemma requires that the multiplier matrix Λ is nonsingular such that both Λ and its inverse Λ^{-1} exist and can be partitioned as

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix}, \qquad \Lambda^{-1} = \Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^T & \Psi_{22} \end{bmatrix} \tag{24}$$

Applying the dualization lemma to the primal SDP (23), with K defined in Eq. (21), results in

$$\begin{bmatrix} A_l^T & C_l^T \\ B_l^T & D_l^T \\ -I & 0 \\ 0 & -I \end{bmatrix}^T K^{-1} \begin{bmatrix} A_l^T & C_l^T \\ B_l^T & D_l^T \\ -I & 0 \\ 0 & -I \end{bmatrix} \succ 0$$
$$\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & -\mathbf{\Delta^T} \end{bmatrix}^T K^{-1} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & -\mathbf{\Delta^T} \end{bmatrix} \preceq 0 \quad \forall\ \mathbf{\Delta} \in \mathcal{S} \tag{25}$$

The first matrix inequality in Eq. (25), denoted as the dual SDP, is obtained by inserting the explicit expression for K^{-1}, resulting in

$$\begin{bmatrix} A_l^T & C_l^T \\ B_l^T & D_l^T \\ -I & 0 \\ 0 & -I \end{bmatrix}^T \overbrace{\begin{bmatrix} -\frac{1}{\gamma} & 0 & 0 & 0 \\ 0 & \Psi_{11} & 0 & \Psi_{12} \\ 0 & 0 & \gamma I & 0 \\ 0 & \Psi_{12}^T & 0 & \Psi_{22} \end{bmatrix}}^{K^{-1}} \begin{bmatrix} A_l^T & C_l^T \\ B_l^T & D_l^T \\ -I & 0 \\ 0 & -I \end{bmatrix}$$
$$= \begin{bmatrix} B_l\Psi_{11}B_l^T + \gamma I & B_l(\Psi_{11}D_l^T - \Psi_{12}) \\ (D_l\Psi_{11} - \Psi_{12}^T)B_l^T & \Psi_{22} - D_l\Psi_{12} - \Psi_{12}^T D_l^T + D_l\Psi_{11}D_l^T \end{bmatrix}$$
$$-\frac{1}{\gamma}\begin{bmatrix} A_l \\ C_l \end{bmatrix} \cdot \begin{bmatrix} A_l \\ C_l \end{bmatrix}^T \succ 0 \tag{26}$$

This matrix inequality is not a LMI, since it is nonlinear in the variable γ and in the decision variables $\mathbf{u}$. However, by taking the Schur complement, one can obtain the desired LMI form of the dual SDP. A matrix-valued mapping T is introduced, similar to the primal constraint (22), such that the set of all multipliers Ψ for which $T(\Psi)$ is negative-definite is a subset of all multipliers that satisfy Eq. (25), yielding

$$T(\Psi) \prec 0 \Rightarrow \begin{bmatrix} I \\ -\mathbf{\Delta}^T \end{bmatrix}^T \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^T & \Psi_{22} \end{bmatrix} \begin{bmatrix} I \\ -\mathbf{\Delta}^T \end{bmatrix} \preceq 0 \quad \forall\ \mathbf{\Delta} \in \mathcal{S} \tag{27}$$

Hence, the condition imposed on $T(\Psi)$ implies that the second matrix inequality in Eq. (25) is satisfied for all $\mathbf{\Delta} \in \mathcal{S}$. The same multipliers as in Sec. IV.B can be applied and similar conditions can be derived for the dual multiplier formulation.

V. Robust Min-Max Strategies

A. LFR Reformulation

The previously defined LFR (16) can easily be relaxed by the S-procedure in the case of open-loop controls ($\mathbf{M} = 0$). However, once the control input class is extended to contain the affine disturbance-feedback strategies, a dependence on the feedback gains $\mathbf{M}$ is introduced into B_l and D_l. This dependency will highly complicate the problem, as both the primal Eq. (21) and dual Eq. (26) are not convex. Nevertheless, since the low-order LFR (16) is nonunique, it can be refactorized such that all decision variables are located in the A_h and C_h blocks of the new high-order LFR. Note that B_l and D_l can be partitioned as $B_l = [\,B_{ld} \quad B_s \quad B_n\,]$ and $D_l = [\,D_{ld} \quad D_s \quad D_n\,]$.

Let us now perform a scalar decomposition of $\mathbf{d}$ and a similar decomposition of $\mathbf{M}$ in its $n_2 = n_c \cdot n_d$ column vectors $m_j \in \mathbb{R}^{n_c \cdot n_u}$ in such a fashion that

$$B_{ld}(\mathbf{M})\mathbf{\Delta}_{ld}(\mathbf{d})C_{ld} = \sum_{j=1}^{n_2}\left(\begin{bmatrix} \mathbf{Q}^{1/2}\mathbf{G}_d \\ 0 \end{bmatrix}_j + \begin{bmatrix} \mathbf{Q}^{1/2}\mathbf{B} \\ \mathbf{R}^{1/2} \end{bmatrix} m_j\right)\mathbf{d}_j$$
$$= \left[\begin{bmatrix} \mathbf{Q}^{1/2}\mathbf{G}_d \\ 0 \end{bmatrix}_1 \quad \begin{bmatrix} \mathbf{Q}^{1/2}\mathbf{B} \\ \mathbf{R}^{1/2} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{Q}^{1/2}\mathbf{G}_d \\ 0 \end{bmatrix}_{n_2} \quad \begin{bmatrix} \mathbf{Q}^{1/2}\mathbf{B} \\ \mathbf{R}^{1/2} \end{bmatrix}\right]$$
$$\times \begin{bmatrix} \mathbf{d}_1 I & & 0 \\ & \ddots & \\ & & \mathbf{d}_{n_2} I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \\ m_1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 1 \\ m_{n_2} \end{bmatrix} \end{bmatrix} = B_{hd}\mathbf{\Delta}_{hd}(\mathbf{d})C_{hd}(\mathbf{M}) \tag{28}$$

This new refactorization is far from minimal, even after reducing the order of the LFR by eliminating all zeros in the $C_{hd}(\mathbf{M})$ block. Note, furthermore, that $\Delta_{hd}(\mathbf{d})$ has a repeated scalar-diagonal structure. A similar refactorization approach can be applied to the D_{ld} blocks, which finally results in the following high-order LFR:

$$\begin{bmatrix} \mathbf{z}_p \\ \mathbf{z}_h \end{bmatrix} = \begin{bmatrix} A_h(x_0, \mathbf{v}) & B_h \\ C_h(x_0, \mathbf{v}, \mathbf{M}) & D_h \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{w}_h \end{bmatrix}, \qquad \mathbf{w}_h = \mathbf{\Delta}_h(\boldsymbol{\delta})\mathbf{z}_h \tag{29}$$

with $\mathbf{\Delta}_h(\boldsymbol{\delta}) = \mathrm{diag}(\mathbf{\Delta}_{hd}(\mathbf{d}), \mathbf{\Delta}_s(\boldsymbol{\delta}_s), \mathbf{\Delta}_n(\boldsymbol{\delta}_n))$.

B. Main Linear Matrix Inequality

The direct verification of the semi-infinite state and control constraints in Eq. (18) involves infinitely many LMI conditions, and is therefore computationally intractable. Lagrange relaxations can be applied as a tractable method that guarantees robust satisfaction of the constraints. This robustification of the state and control constraints results in explicit second-order cone programs or SDPs (see Ben-Tal and Nemirovski [16]). The explicit LMIs will not be shown in the sequel, since these robust constraints are handled in the same way as the performance constraint.

The main SDP to obtain the robust disturbance feedback [Eq. (17)] is defined by the LMIs summarized in the following theorem.

Theorem 3: The robust disturbance-feedback strategy $\mathbf{u} = \mathbf{v}^* + \mathbf{M}^*\mathbf{d}$ applied to the system represented by LFR (29) guarantees an upper bound γ^* on the performance in Eq. (19) if $\gamma^*, \mathbf{v}^*, \mathbf{M}^*, \Psi^*$ are a solution to the following SDP problem:

$$\inf_{\gamma,\mathbf{v},\mathbf{M},\Psi} \quad \gamma \quad \text{subject to} \begin{bmatrix} \gamma & A_h(x_0,\mathbf{v})^T & C_h(x_0,\mathbf{v},\mathbf{M})^T \\ A_h(x_0,\mathbf{v}) & B_h\Psi_{11}B_h^T + \gamma I & B_h(\Psi_{11}D_h^T - \Psi_{12}) \\ C_h(x_0,\mathbf{v},\mathbf{M}) & (D_h\Psi_{11} - \Psi_{12}^T)B_h^T & \Psi_{22} - D_h\Psi_{12} - \Psi_{12}^T D_h^T + D_h\Psi_{11}D_h^T \end{bmatrix} \succ 0$$
$$E(\mathbf{v} + \mathbf{M}\mathbf{d}) \preceq 0 \quad \forall\ \mathbf{\Delta} \in \mathcal{S}, \qquad F(\mathbf{x}) \preceq 0 \quad \forall\ \mathbf{\Delta} \in \mathcal{S}, \qquad T(\Psi) \prec 0 \tag{30}$$

Proof. The proof flows trivially by applying the full-block S-procedure to the problem defined by Eqs. (15), (17), (18), (27), and (29). □

The SDP formulation in Eq. (30) has the advantage that it allows for models with a mixture of different uncertainty types. The refactorization of the LFR does not lead to any additional conservatism due to the extension of the multiplier class to full-block multipliers.

C. Receding-Horizon Implementation

Assume that the finite-horizon discrete-time problem has a problem horizon n_f with time instants $i \in \mathcal{I} = \{0, \ldots, n_f\}$. One way to obtain a guaranteed performance for the whole intercept is to take the control horizon n_c the same as the problem horizon; hence, $n_f = n_c$ and $\mathcal{I} = \mathcal{K}$. However, in the case of large problem horizons this approach leads to large computational demands and potential conservatism. The performance can be improved by implementing the robust strategies as receding-horizon control [5] strategies. The basic principle of RHC is to determine and implement a new strategy at each time step $i \in \mathcal{I}$ i.e., a robust strategy $\mathbf{u}(x(i))$ with initial condition $x(i)$ is obtained for all time instants on the control horizon $k \in \mathcal{K}$ and the first control action $u(0) = v(0) \in \mathbf{u}(x(i))$ of this strategy is applied. Recall that $u(x_0)$ depends on x_0, although this has been dropped until now for notational convenience. The admissible robust RHC [20] strategy $\mu\colon \mathcal{I} \times \mathcal{X}(\mathcal{I}) \to \mathcal{U}(\mathcal{I})$ is defined as

$$\begin{aligned}\mu(i, x) \triangleq \{&v(0) \in \mathbb{R}^{n_u} | v(0) \in \mathbf{u}(x)\\ &\mathbf{u}(x) \text{ solution SDP in Eq. (30)}\}\end{aligned} \tag{31}$$

The robust RHC strategy allows to adapt the robust disturbance-feedback (RDF) strategy at each time instant $i \in \mathcal{I}$ and use the state $x(i)$, which is assumed available, as the initial state for a new robust disturbance-feedback strategy.

VI. Illustrative Intercept Example

In the following example, a missile intercept in the homing phase is considered. The problem formulation of the intercept, with its dynamics described in Sec. II, exactly matches the previous given problem definition in Sec. III.

A. Simulation Approach

The simulation approach is to generate a set with some relevant test scenarios and simulate the performance of the guidance laws on these test cases. The test cases introduced in Ben-Asher and Yaesh [1] represent the most relevant disturbances in a missile intercept and include step, bang–bang and sinusoidal evader acceleration, and initial heading errors. The scenarios have proved to be an excellent test-bench for the differential-game-based guidance laws and a representative selection is applied to this example. Moreover, it is assumed that all system uncertainties δ_t and δ_ω and disturbance d are varying in time.

The performance of the RHC implementation of the robust disturbance-feedback strategies (RDF) will be compared with three nominal and one robust strategy. The nominal strategies are the traditional augmented proportional navigation law (APN), the soft-constrained minimum-effort law (MEL) [1], and the hard-constrained bang–bang law (BGL) [2]. The robust guidance law (ROB) [3] is derived by including the energy of the uncertainty signal in the cost function. The problem is reformulated as an auxiliary disturbance problem with a completely known system. The approach is based on finite-time-horizon H_∞-optimal control for real bounded time-varying uncertainties [21]. The weight parameters for MEL ($\gamma \to \infty$, $b = 1000$) and ROB ($\gamma = 4.2$, $b = 1000$, $\epsilon = 1.5$) have been taken identical to the values in Ben-Asher and Yaesh [1]. Roughly speaking, the variable b is the penalty weight on the final miss distance, γ is the penalty weight on the target maneuvers, and ϵ is a scaling variable to prevent input saturation. The exact definition of these parameters can be found in the references.

The intercept parameters are chosen such that it allows for a fair comparison with examples from the literature. The target time constant is assumed to be contained in the interval [0.2,0.5] s, yielding $\tau_e = \frac{2}{7}\ \mathrm{s}^{-1}$ and $|\delta_e(t)| \leq 1.5\ \mathrm{s}^{-1}$. The nominal actuator dynamics are characterized by $\omega = 2.5\ \mathrm{rad} \cdot \mathrm{s}^{-1}$ and $\zeta = 0.65$. The parameters in the nonlinear mapping [Eq. (32)] are taken as $c_0 = 46\ \mathrm{g} \cdot \mathrm{rad}^{-1}$ and $c_1 = -60\ \mathrm{g} \cdot \mathrm{rad}^{-3}$ leading to a maximum lateral interceptor acceleration of ± 15 g. It is assumed that the control effort is cheap and that the objective is to track the collision course, resulting in the following weight matrices $R(\cdot) = 0.01$, $Q(\cdot) = 0$ $\forall\ k \in \mathcal{K}/\{n_c\}$, and $Q(n_c) = [1 \quad 0 \quad 0 \quad 0]$. The control, state, and uncertainty sets are given as

$$\begin{aligned}\mathcal{X}_3(k) &= \{x_3(k) \in \mathbb{R} | \quad |x_3(k)| \leq 25\ \deg\}\\ \mathcal{U}(k) &= \{u(k) \in \mathbb{R} | \quad |u(k)| \leq 25\ \deg\}\\ \mathcal{D}_d(k) &= \{d(k) \in \mathbb{R} | \quad |d(k)| \leq 3\ \mathrm{g}\}\\ \mathcal{D}_s(k) &= \{\delta(k) \in \mathbb{R}^2 | \quad |\delta_e(k)| \leq 1.5\ \mathrm{s}^{-1} \quad |\delta_\omega(k)| \leq 0.25\}\end{aligned}$$

The intercept duration is taken as $t_f = 3.0$ s and all guidance laws have been discretized using a zero-hold operation and a sampling time of $t_d = 0.1$ s.

The robust approach discussed in this paper requires a LFR of the nonlinear continuous system as is done in Sec. III. The variables are defined as the state $x = \mathrm{col}(r, v, \phi, \dot{\phi}, a_t)$, input $u = \phi_c$, disturbance $d = a_{t,c}$ and uncertainty $\delta_s = \mathrm{col}(\delta_\omega, \delta_e)$, which results after combining Eqs. (1–5), in the following LFR:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & c_0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & -\omega^2 & -2\zeta\omega & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{\tau_e} \end{bmatrix} x(t) + \begin{bmatrix} 0\\ 0\\ 0\\ \omega^2\\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ \frac{1}{\tau_e} & 0 & 0 & 1 & 0 \end{bmatrix} w(t)$$

with auxiliary uncertainty signal $w(t) = \mathrm{col}(w_d(t), w_s(t), w_n(t))$ and $z(t) = \mathrm{col}(z_d(t), z_s(t), z_n(t))$ as well as the output equation:

$$z(t) = \begin{bmatrix} 1\\ 0\\ 0\\ 0\\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -2\omega^2 & -2\zeta\omega & 0\\ 0 & 0 & -\omega^2 & 0 & 0\\ 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0\\ 2\omega^2\\ \omega^2\\ 0\\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} w(t)$$

Note that for the derivation of the LFR the auxiliary uncertainty signals w_n and z_n have been introduced to cover the model nonlinearity. The nonlinear mappings $g_i(x, u)$ are characterized as $g_i(x, u) = 0$, for $i = 1, 3, 4, 5$ and

$$g_2(x, u) = c_1 x_3^3 \tag{32}$$

The pair (w_n, z_n) is contained in the conic sector with $z_n = x_3$, $w_n \in \{\delta_n z_n | \delta_n \in [\beta, \sigma]\}$ and $\beta = c_1 x_{3,\max}^2$, $\sigma = 0$. The LMIs (30) in this example are solved in MATLAB®, used along with the parser YALMIP [22].

B. Simulation Results

The discretized guidance laws have been applied on the continuous model as defined by Eqs. (1–5). The traditional guidance

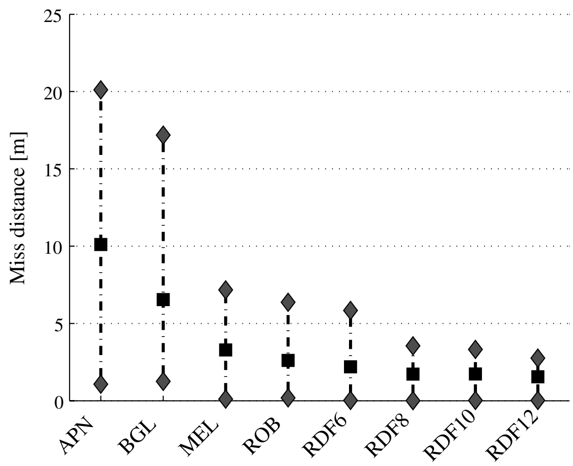
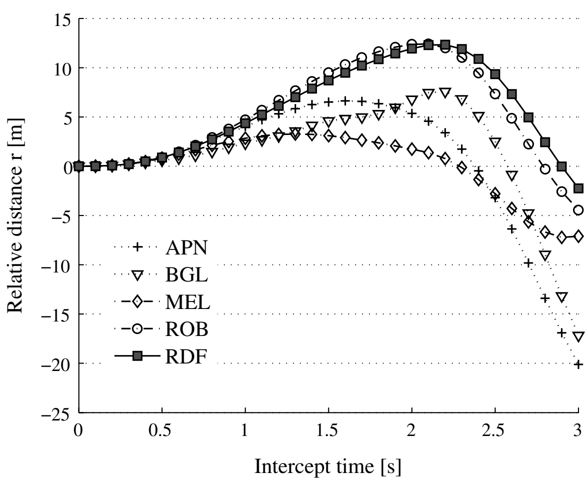
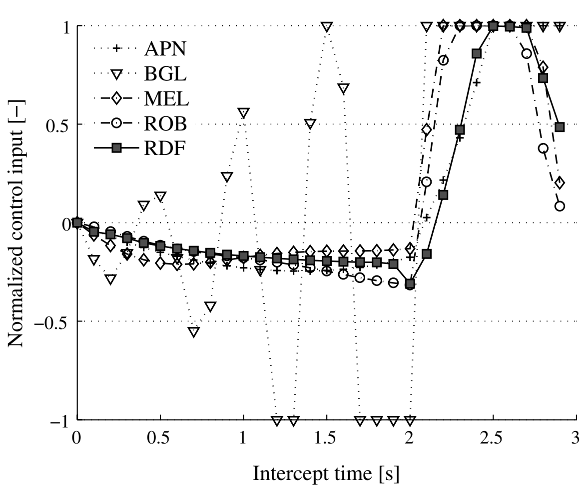


Fig. 3 Different miss-distance guidance laws: □ average and ◇ min or max.



a)



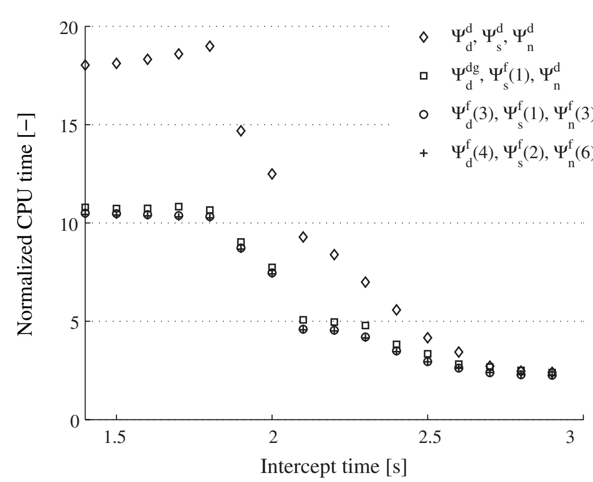
b)

Fig. 4 GLs characteristics, uncertain model, 3 *g* bang–bang target maneuver.

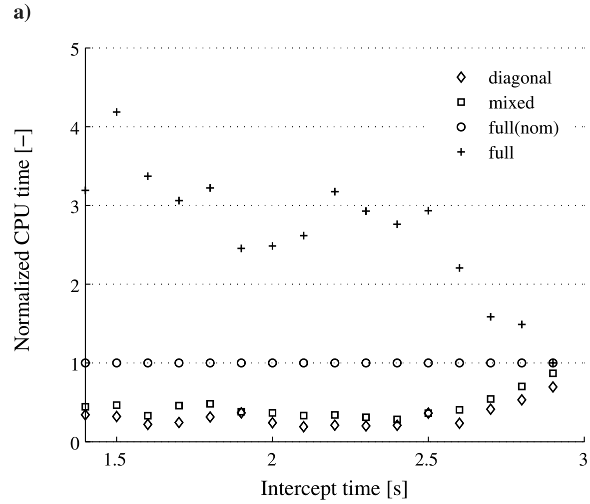
laws (APN, BGL, MEL, and ROB) generate a commanded acceleration $a_{m,c}$ instead of the required model input, which is the commanded tail fin deflection ϕ_c. It is now assumed that the traditional guidance laws are implemented by neglecting higher-order dynamics and only considering the linear dependence between $a_{m,c}$ and ϕ_c defined as $\phi_c = c_0^{-1} a_{m,c}$.

Figure 3 clearly shows that the miss-distance performance of RDF, for control horizon $n_c \geq 6$, is superior to all other guidance laws on the considered set of disturbances and uncertainties. RDF also outperforms the traditional guidance laws when comparing the average performances. Simulation results have shown that using a control horizon $n_c > 12$ or disturbance-feedback horizon $n_m > 1$ is not beneficial to performance and only leads to a higher computational load. Hence, a baseline RDF with $n_c = 12$ and $n_m = 1$ is taken in the next section. An illustration of the RDF characteristics is supplied by taking the disturbance and uncertainty realization that leads to the largest miss distance for RDF. The considered bang–bang trajectories are $\delta_e(t) = -1.5\ \mathrm{s}^{-1}$, $\delta_\omega(t) = 0.25$ and $d(t) = 3$ g for $t \leq 2$ s and $\delta_e(t) = 1.5\ \mathrm{s}^{-1}$, $\delta_\omega(t) = -0.25$ and $d(t) = -3$ g for $t > 2$ s. Figure 4 shows the characteristics of the different guidance laws in the case of these bang–bang realizations.

In the previous section it is claimed that block-diagonal Ψ^{dg} and full-block multipliers Ψ^f lead to better guaranteed performance than the traditional scalar-diagonal multipliers Ψ^d, because $\Psi^d \subseteq \Psi^{dg} \subseteq \Psi^f$. This phenomenon is illustrated by applying different multipliers to the final phase of the previous example. At each discrete-time instant a disturbance-feedback problem, with a different choice of multipliers, is solved by taking the state at time $i \in \mathcal{I}$ of the example state trajectory as initial state, i.e., $x_0 = x(i)$. The used multipliers are partitioned similar as $\boldsymbol{\Delta} = \mathrm{diag}(\boldsymbol{\Delta}_d, \boldsymbol{\Delta}_s, \boldsymbol{\Delta}_n)$, resulting in $\Psi = \mathrm{diag}(\Psi_d, \Psi_s, \Psi_n)$. However, the problem still becomes computational intractable when applying larger control horizons and full-block multipliers. A potential remedy is to no longer consider full-block multipliers for the complete lifted blocks $\boldsymbol{\Delta}_d$, $\boldsymbol{\Delta}_s$ and $\boldsymbol{\Delta}_n$ but to form full diagonal



a)



b)

Fig. 5 Multipliers' performances.

blocks. Hence, full-block multipliers are only applied to subblocks of $\mathbf{\Delta}_d$, $\mathbf{\Delta}_s$ and $\mathbf{\Delta}_n$ leading to a similar additional partitioning of the multipliers. The superscript of the full-block-diagonal multipliers denotes the number of diagonal blocks that are grouped together into one subblock. In other words, the notation Ψ_s^{fi} means that i delta blocks $\Delta(i)$ are grouped together, while the dimension of the last subblock is chosen such that the new decomposition matches the original one, e.g., assuming $n_c = 10$ the notation Ψ_s^{f4} means that the multipliers consist of full blocks, respectively, relaxing $\mathrm{diag}(\Delta_s(\delta(0)), \ldots, \Delta_s(\delta(4)))$, $\mathrm{diag}(\Delta_s(\delta(5)), \ldots, \Delta_s(\delta(8)))$, and the last remaining subblock $\mathrm{diag}(\Delta_s(\delta(9)), \Delta_s(\delta(10)))$. The baseline multipliers, referred to as full (nom), is the full-block-diagonal multiplier defined as $(\Psi_d^{f3}, \Psi_s^{f1}, \Psi_n^{f3})$. The other multiplier structures that are compared with each other, on the worst-case trajectories, are diagonal $(\Psi_d^{d}, \Psi_s^{d}, \Psi_n^{d})$, mixed $(\Psi_d^{\mathrm{dg}}, \Psi_s^{f1}, \Psi_n^{d})$, and full $(\Psi_d^{f4}, \Psi_s^{f2}, \Psi_n^{f6})$ multipliers. Figure 5a clearly shows the increase in performance when better inner approximations of the multiplier set are used. The disadvantage of better inner approximations is the increase in computational load depicted in Fig. 5b. Note that the CPU times are normalized with respect to the CPU time of the baseline RDF. The introduction of the full-block multipliers hence allows for a tradeoff between performance and computational load. The example shows that only little performance can be gained by using superior multipliers instead of the employed nominal multipliers while the computational load substantially increases. Figure 5a also clearly demonstrates that, once the problem and control horizon coincide ($t \geq 1.8$ s), the performance upper bound γ is monotonically nonincreasing.

The high-order LFR (29) is introduced such that robust disturbance-feedback strategies can be computed for nonlinear systems with a rational uncertainty dependence and additive disturbances. The Frobenius norm of $\mathbf{M}$, denoted as $\|\mathbf{M}\|_F$, is chosen as a measure for the level of disturbance feedback of the robust strategies. Figure 6 depicts that the restriction of the multiplier structure to the traditional scalar-diagonal structures leads to $\|\mathbf{M}\|_F = 0$ and hence to the equivalence between robust disturbance-feedback and open-loop strategies. The consequence of the lack of disturbance feedback is the conservative performance, as shown in Fig. 5a. Figure 6, furthermore, demonstrates that the extension of the multiplier class to full-block multipliers increases the level of disturbance feedback and as such increases the performance of RDF.

This illustrative example clearly shows that implementing robust RHC strategies can be highly effective in reducing the miss distance. The robust approach is generic and requires only that the uncertainties and disturbances are contained in a priori known sets. Moreover, the robust problem formulation effectively deals with hard constraints imposed on the state and/or control. The robust framework allows to tackle large classes of model uncertainties with the same ease as in the simple case used in this paper. Although superior performances have been obtained by the robust disturbance-feedback strategies one might have expected more improvement of such a complex guidance law design. In the first place it is important to realize that the guarantee of a zero miss distance is not possible in the case of persistent disturbances and uncertainties. Hence, instead of comparing miss distances, one would ideally like to compare the deviations in miss distance from the optimal upper bound of the closed-loop system, which is, unfortunately, not possible. Second, it is expected that the full-block robust disturbance-feedback strategies will be even more effective in dealing with, and hence exploiting, more complicated uncertainty structures.

A large drawback of the robust strategies is their computational load, which highly complicates the online implementation of such strategies. However, more heuristic approaches can be followed here. For example, one can design the feedback gains $\mathbf{M}$ offline using the effective but computationally demanding multipliers. These gains can then be stored and implemented with a gain scheduling approach. Online implementation of the resulting robust open-loop min-max strategies $\mathbf{v}$ can then be computed by using multipliers that require less computational effort, such as the traditional diagonal multipliers. This approach is already suggested in previous studies in the case of a static state-feedback gain [23]. Another possibility is to exploit the LMI structure of the problem to design parameter-dependent controllers.

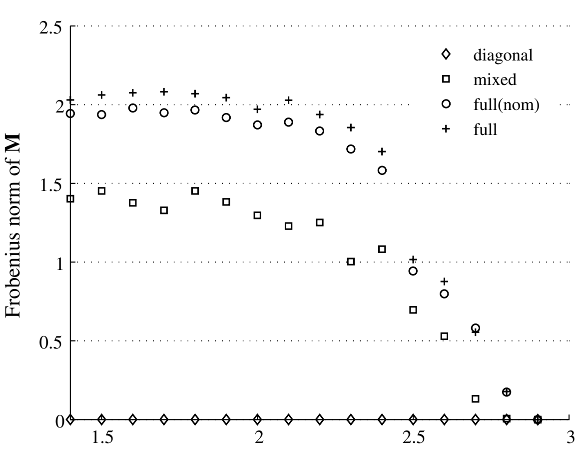


Fig. 6 Disturbance-feedback gains.

VII. Conclusions

This paper introduced a class of receding-horizon guidance laws based on a robust control design approach developed by Lofberg [6]. It was shown that robust guidance algorithms can be derived based on well-understood numerical solutions of linear matrix inequalities. The major contribution of this study is the generalization of previous work by the following:

1) Extend the class of multipliers to the robust disturbance-feedback strategies with full-block multipliers possibly reducing the conservatism introduced by the relaxation techniques.

2) Generalize the class of robust disturbance-feedback strategies by making the framework suitable to incorporate mixed uncertainties containing additive polytopic, rational, and sector-bounded uncertainties.

The proposed guidance laws were illustrated on a simple intercept model and compared to other robust guidance laws available in the literature [2,3]. With the help of numerical simulations, it was shown that our approach leads to superior designs in terms of guaranteed miss-distance bounds. It has thus been shown that robust programming can successfully be applied to intercept problems. Furthermore, it allows to tackle a large class of model uncertainties in the guidance loop with the same ease as in the simple case used in this paper. However, this improvement is obtained at the price of increased computational complexity. Therefore, future research will focus on exploiting the LMI problem structure to transfer the computationally demanding controller design offline and implement it online as a parameter-dependent controller.

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